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Hamilton-Connected Cayley Graphs on Hamiltonian Groups

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It is proven that every connected Cayley graph X, of valency at least three, on a Hamiltonian group is either Hamilton laceable when X is bipartite, or Hamilton connected when X is not bipartite.

1. INTRODUCTION

Graphs in this paper have neither loops nor multiple edges and are finite. If X is a regular graph of valency r, we shall denote this by $\mathrm{val}(X) = r$. We shall be dealing with Hamilton paths and cycles in a particular family of Cayley graphs.

DEFINITION 1.1. Let G be a finite group and S a subset of G satisfying $1 \notin S$ and $s \in S$ if and only if $s^{-1} \in S$. The *Cayley graph* $X = X(G; S)$ is a graph with vertex set G and $xy \in E(X)$ if and only if $y = xs$ for some $s \in S$. Note that X is connected if and only if S is a generating set of G.

There has been considerable work dealing with the following conjecture. (For a recent survey see [5].)

CONJECTURE 1.2. *Every connected Cayley graph with more than two vertices has a Hamilton cycle.*

The first family of graphs for which Conjecture 1.2 was established was the family of Cayley graphs on abelian groups. It was independently proven by several people and is included in [7]. In fact, a much stronger result about the Hamiltonicity of Cayley graphs on abelian groups is known. It follows the next definition.

DEFINITION 1.3. A graph is said to be *Hamilton-connected* if for any two vertices there exists a Hamilton path joining them. Analogously, a bipartite graph with bipartition sets A and B satisfying $|A| = |B|$ is said to be *Hamilton-laceable* if for any $u \in A$ and $v \in B$ there is a Hamilton path joining u and v.

THEOREM 1.4 ([3]). *Let X be a connected Cayley graph on an abelian group. If* $\mathrm{val}(X) = 2$, *X is a cycle. If* $\mathrm{val}(X) > 2$, *then*

(i) *X is Hamilton-connected if X is not bipartite;*
(ii) *X is Hamilton-laceable if X is bipartite.*

We refer to the preceding theorem as the Chen–Quimpo theorem throughout the paper. Are there other families of groups which admit analogues of the Chen–Quimpo theorem? A natural direction in which to look is towards groups that are, in some sense, 'almost' abelian. The dihedral groups have been investigated [2]. Another family of groups, and the subject of this paper, is the family of Hamiltonian groups.

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DEFINITION 1.5. A finite non-abelian group G in which every subgroup is normal is called a *Hamiltonian group*.

Chen and Quimpo [4] investigated Cayley graphs on Hamiltonian groups and proved the following result.

THEOREM 1.6. *Let G be a Hamiltonian group and S a generating set of G. Then $X(G; S)$ has a Hamilton cycle.*

In this paper we prove that the analogue of the Chen–Quimpo theorem holds for Cayley graphs on Hamiltonian groups. We shall find that generalized Petersen graphs play a special role in the proof of the main theorem.

2. A GENERALIZED PETERSEN GRAPH

DEFINITION 2.1. The *generalized Petersen graph $GP(n, k)$*, $n \geq 2$ and $1 \leq k \leq n-1$ has vertex set

$$\{x_0, x_1, \dots, x_{n-1}, y_0, y_1, \dots, y_{n-1}\}$$

and edge set

$$\{x_i x_{i+1}, x_i y_i, x_i x_{i+k} : 0 \leq i \leq n-1 \text{ with subscripts reduced modulo } n\}.$$

The first author has asked whether the generalized Petersen graph $GP(n, k)$, where gcd $(n, k) = 1$ and $GP(n, k)$ is not isomorphic to $GP(6m+5, 2)$ for some integer m, is Hamilton-connected or Hamilton-laceable. In this paper we prove that $GP(4m, 2m-1)$ is Hamilton-laceable and use this result in the proof of the main theorem.

There are several results about generalized Petersen graphs that we also shall employ. They are now stated followed by Theorem 2.5 and its proof.

PROPOSITION 2.2 ([6]). *If $GP(n, k)$ is Hamiltonian, it is edge-Hamiltonian; that is, each edge of $GP(n, k)$ is on a Hamilton cycle.*

PROPOSITION 2.3 ([6]). *The generalized Petersen graph $GP(n, k)$ is bipartite if and only if n is even and k is odd.*

THEOREM 2.4 ([1]). *The generalized Petersen graph $GP(n, k)$ is Hamiltonian if and only if it is neither*

(i) $GP(n, k) \cong GP(n, 2) \cong GP(n, n-2) \cong GP(n, (n-1)/2) \cong GP(n, n+1/2)$, $n \equiv 5 (\mathrm{mod}\, 6)$, *nor*
(ii) $GP(n, k) \cong GP(n, n/2)$, $n \equiv 0 (\mathrm{mod}\, 4)$ *and* $n \geq 8$.

THEOREM 2.5. *$GP(4m, 2m-1)$ is Hamilton-laceable.*

PROOF. We know $GP(4m, 2m-1)$ is bipartite by Proposition 2.3. We can relabel the vertices of $GP(4m, 2m-1)$ as follows: y_i becomes v_i, $i = 0, 1, \dots, 4m-1$, x_{2j} becomes u_{2j}, $0 \leq j \leq 2m-1$, and $x_{2m+2j-1}$ becomes u_{2j-1}, $1 \leq j \leq 2m$, where subscripts are calculated modulo $4m$. Under this relabelling, the vertex set is

$$\{u_0, u_1, \dots, u_{4m-1}, v_0, v_1, \dots, v_{4m-1}\}$$

and the edge set is

$$\{u_{2i}v_{2i},\, u_{2i+1}v_{2m+2i+1},\, v_{2i+1}u_{2m+2i+1},\, u_ju_{j+1},\, v_jv_{j+1} : 0 \le i \le 2m-1,\, 0 \le j \le 4m-1\}.$$

By Proposition 2.2, Theorem 2.4 and symmetry, it suffices to prove that there is a Hamilton path from u_0 to each vertex v of $\{u_3, u_5, \ldots, u_{2m-1}, v_2, v_4, \ldots, v_{2m}\}$. There are five cases to consider.

CASE 1: $v \in \{v_2, v_4, \ldots, v_{2m-4}\}$, say $v = v_{2i}$. We list three paths of $GP(4m, 2m-1)$ as follows. Let

$$\begin{aligned}
P_1 &= u_0u_{4m-1}u_{4m-2}v_{4m-2}v_{4m-3}u_{2m-3}u_{2m-2}u_{2m-1}v_{4m-1}v_0v_1\\
&\quad \cdots v_{2i-1}u_{2m+2i-1}u_{2m+2i-2}u_{2m+2i-3}\cdots u_{2m}v_{2m}v_{2m-1}v_{2m-2}v_{2m-3},\\
P_2 &= v_{2i}u_{2i}u_{2i-1}u_{2i-2}\cdots u_1v_{2m+1}v_{2m+2}\cdots v_{2m+2i}, \qquad \text{and}\\
P_3 &= v_{2m+2i}u_{2m+2i}u_{2m+2i+1}v_{2i+1}v_{2i+2}u_{2i+2}u_{2i+1}v_{2m+2i+1}v_{2m+2i+2}\\
&\quad u_{2m+2i+2}u_{2m+2i+3}v_{2i+3}v_{2i+4}u_{2i+4}u_{2i+3}v_{2m+2i+3}v_{2m+2i+4}\\
&\quad u_{2m+2i+4}u_{2m+2i+5}v_{2i+5}\cdots v_{2m-3}.
\end{aligned}$$

Because $(2m-3)-(2i+1) = 2m-2i-4 = (4m-4)-(2m+2i)$, the end vertex of P_3 is exactly v_{2m-3}. It is easy to see that P_1, P_2 and P_3 are pairwise vertex-disjoint except for their end vertices, and that their union is a Hamilton path from u_0 to v.

CASE 2: $v = v_{2m-2}$. We have the following Hamilton path P from u_0 to v:

$$\begin{aligned}
P &= u_0u_{4m-1}v_{2m-1}v_{2m}u_{2m}u_{2m-1}v_{4m-1}v_0v_1u_{2m+1}u_{2m+2}v_{2m+2}v_{2m+1}u_1u_2\\
&\quad v_2v_3u_{2m+3}u_{2m+4}v_{2m+4}v_{2m+3}u_3u_4v_4v_5u_{2m+5}\cdots u_{2m-3}u_{2m-2}v_{2m-2}.
\end{aligned}$$

CASE 3: $v = v_{2m}$. We construct a Hamilton path P as follows:

$$\begin{aligned}
P &= u_0u_{4m-1}v_{2m-1}v_{2m-2}v_{2m-3}\cdots v_2u_2u_1v_{2m+1}v_{2m+2}u_{2m+2}u_{2m+3}u_{2m+4}\\
&\quad \cdots u_{4m-2}v_{4m-2}v_{4m-3}v_{4m-4}\cdots v_{2m+3}u_3u_4u_5\cdots u_{2m-1}v_{4m-1}v_0v_1\\
&\quad u_{2m+1}u_{2m}v_{2m}.
\end{aligned}$$

CASE 4: $v \in \{u_3, u_5, \ldots, u_{2m-3}\}$. Let $v = u_{2i+1}$. We construct a Hamilton path by using three subpaths as in Case 1. Let

$$\begin{aligned}
P_1 &= u_0v_0v_{4m-1}v_{4m-2}u_{4m-2}u_{4m-1}v_{2m-1}v_{2m}v_{2m+1}u_1u_2u_3\cdots u_{2i}v_{2i}v_{2i-1}\\
&\quad v_{2i-2}v_{2i-3}\cdots v_1u_{2m+1}u_{2m}u_{2m-1}u_{2m-2}v_{2m-2},\\
P_2 &= u_{2i+1}v_{2m+2i+1}v_{2m+2i}v_{2m+2i-1}\cdots v_{2m+2}u_{2m+2}u_{2m+3}u_{2m+4}\\
&\quad \cdots u_{2m+2i+1}, \qquad \text{and}\\
P_3 &= u_{2m+2i+1}v_{2i+1}v_{2i+2}u_{2i+2}u_{2i+3}v_{2m+2i+3}v_{2m+2i+2}u_{2m+2i+2}u_{2m+2i+3}v_{2i+3}\\
&\quad v_{2i+4}u_{2i+4}u_{2i+5}v_{2m+2i+5}v_{2m+2i+4}u_{2m+2i+4}\cdots u_{4m-3}v_{2m-3}v_{2m-2}.
\end{aligned}$$

Since $(4m-3)-(2m+2i+1) = (2m-3)-(2i+1)$, P_3 consists of all the vertices but those in $P_1 \cup P_2$ and these three subpaths are vertex-disjoint other than their end vertices. So $P_1 \cup P_2 \cup P_3$ is a Hamilton path from u_0 to u_{2i+1}.

CASE 5: $v = u_{2m-1}$. We have the following Hamilton path P:

$$\begin{aligned}
P &= u_0v_0v_{4m-1}v_{4m-2}\cdots v_{2m+2}u_{2m+2}u_{2m+3}u_{2m+4}\cdots u_{4m-1}v_{2m-1}v_{2m}\\
&\quad v_{2m+1}u_1u_2\cdots u_{2m-2}v_{2m-2}v_{2m-3}v_{2m-4}\cdots v_1u_{2m+1}u_{2m}u_{2m-1}.
\end{aligned}$$

This completes the proof. □

We mention that Nedela and Škoviera [8] have proven that $GP(4m-1, 2m-1)$ is a Cayley graph.

3. Cayley Graphs on Hamiltonian Groups

Throughout this section, we let G be a Hamiltonian group. Then we know that $G = Q \times U \times V$, where Q is the quaternion group, U is an odd order abelian group and V is an abelian group of exponent two. Accordingly, we let $G = \{(q, u, v) : q \in Q, u \in U, v \in V\}$. Taking $a = (q, u, v)$ in G, we know

$$o(a) = \begin{cases} o(u) & \text{if } o(q) = o(v) = 1 \\ 2o(u) & \text{if } o(q) = 2, o(v) = 1 \text{ or } o(q) = 1, o(v) = 2 \\ 4o(u) & \text{if } o(q) = 4, o(v) \in \{1, 2\}. \end{cases}$$

If $o(q) \neq 4$, then a is in the centre of G [9, p. 93].

Following are three results which will be useful in our proofs.

THEOREM 3.1 ([4]). *The Cayley graph $X(G; S)$, where G is a Hamiltonian group, is Hamiltonian for any generating set S of G.*

LEMMA 3.2 ([4]). *Let X_i, $i = 1, 2, \dots, N$ be vertex-disjoint Hamilton-connected graphs satisfying $|V(X_i)| = n > 2$ for all i. If there exists a perfect matching F_i between X_i and X_{i+1} for $i = 1, 2, \dots, N-1$, then $\cup_{i=1}^{N} X_i$ together with the edges of the perfect matchings is Hamilton-connected.*

COROLLARY 3.3 ([4]). *Let G be a Hamiltonian group and S a generating set. If S contains an element g with odd order $o(g) > 1$, then $X(G; S)$ is Hamilton-connected.*

LEMMA 3.4. *Let X_i, $i = 1, 2, \dots, N$ be vertex-disjoint Hamilton-laceable bipartite graphs satisfying $|V(X_i)| = 2n > 2$ for all i. If there exists a perfect matching F_i between X_i and X_{i+1} for $i = 1, 2, \dots, N-1$ such that $X = \cup_{i=1}^{N}(X_i \cup F_i) \cup X_N$ is bipartite, then X is Hamilton-laceable.*

PROOF. Let $a \in A$ and $b \in B$, where A and B are the two bipartition sets. Without loss of generality, let $a \in X_i$ and $b \in X_j$ with $i \leq j$. When $i < j$, choose arbitrary vertices $v_i, v_{i+1}, \dots, v_{j-i}$ in B such that $v_t \in X_t$, $i \leq t \leq j-1$. Let $v_i u_{i+1}, v_{i+1} u_{i+2}, \dots, v_{j-1} u_j$ be the edges of $F_i, F_{i+1}, \dots, F_{j-1}$, respectively. Let P_i be a Hamilton path in X_i from a to v_i, P_j be a Hamilton path from u_j to b in X_j and P_{i+r} be a Hamilton path from u_{r+i} to v_{r+i} in X_{i+r} for $1 \leq r \leq j-i-1$. When $j = i$, simply let P_i be a Hamilton path from a to b in X_i. It is easy to see that $P' = P_i \cup P_{i+1} \cup \cdots \cup P_j \cup \{u_{r+i} v_{r+i-1} : 1 \leq r \leq j-i\}$ is a path from a to b containing all the vertices of $X_i \cup X_{i+1} \cup \cdots \cup X_j$.

If $j < N$, let wb be the last edge of P'. Let x_{j+1} and y_{j+1} be the neighbours of w and b, respectively, in X_{j+1} using edges of F_j. Let P_{j+1} be a Hamilton path in X_{j+1} joining x_{j+1} and y_{j+1}. Removing the edge wb from P' and adding the edges of P_{j+1} together with wx_{j+1} and by_{j+1} produces a path from a to b containing all the vertices of $X_i \cup X_{i+1} \cup \cdots \cup X_{j+1}$. We repeat this process using an edge of the path in X_{j+1} until we have a path from a to b using all the vertices of $X_i \cup X_{i+1} \cup \cdots \cup X_N$.

If $i > 1$, choose the edge of P'' incident with a and repeat the above process moving through all X_k, $k < i$. We have constructed a Hamilton path in X from a to b. This completes the proof. □

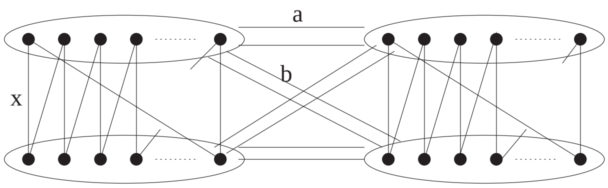

FIGURE 1. $X(G; S)$.

LEMMA 3.5. *Let G be a Hamiltonian group. If S is a generating set of G such that*

(i) $S = \{x, x^{-1}, a, a^{-1}, b, b^{-1}\}$,
(ii) no pair of elements in $\{x, a, b\}$ *commute and*
(iii) $o(x) = o(a) = o(b) = |G|/2$,

then $X = X(G; S)$ *is Hamilton-connected.*

PROOF. Let $I = \langle x^2 \rangle$. Then I is an independent set in X and a normal subgroup in G. By the properties of S, $aI \cap bI = \emptyset$ and $bI = xaI$ (see Figure 1).

Let X' be the spanning subgraph of $X(G; \{a, a^{-1}, x, x^{-1}\})$ obtained by removing all the a-edges from $a\langle x \rangle$ to $\langle x \rangle$ but retaining all the a-edges from $\langle x \rangle$ to $a\langle x \rangle$. Let $o(x) = 4m$.

We claim that $X' \cong GP(4m, 2m-1)$. We know that $xa = ax^i$ for some $1 \le i \le 4m-1$. Since $ax^2 = x^2a = x(xa) = xax^i = ax^{2i}$, $2 \equiv 2i (\mathrm{mod}\, 4m)$, that is $2i - 2 = l(4m)$. Since $i < 4m$, $l \le 1$. If $l = 0$, then $i = 1$, which implies that $ax = xa$ and contradicts G being non-abelian. Thus, $l = 1$ and $i = 2m+1$. Therefore, $X' \cong GP(4m, 2m+1) \cong GP(4m, 2m-1)$.

By Theorem 2.5, X', and hence $X(G; \{a, a^{-1}, x, x^{-1}\})$, is Hamilton-laceable. So for each pair of vertices u and v, with $u \in I \cup xaI$ and $v \in xI \cup aI$, there is a Hamilton path from u to v.

Similarly, $X(G; \{x, x^{-1}, b, b^{-1}\})$ is Hamilton-laceable. So for each vertex $u \in I$ and $v \in bI = xaI$, there is a Hamilton path from u to v. What remains to be proved is that there is a Hamilton path between any pair of vertices $u, v \in I$. For convenience, in the following we relabel X' as follows. Label x^i with u_i and label ax^i with v_i. Then the cycles given by the x-edges are $u_0u_1 \cdots u_{4m-1}$ and $v_0v_1 \cdots v_{4m-1}$. In addition, u_{2i} and v_{2i} are adjacent, and u_{2k+1} and $v_{2k-2m+1}$ are adjacent, where $0 \le i, k \le 2m-1$ and subscripts are reduced modulo $4m$. Then $I = \{u_{2i} : 0 \le i \le 2m-1\}$.

Let u_{4m-1} be adjacent to v_{2l} and v_{2k} by a b-edge and a b^{-1}-edge, respectively. Since $x^2b = bx^2$, u_{2m+1} is adjacent to $v_{2l-2m+2}$ and $v_{2k-2m+2}$. Since $o(b) \ne 2$, we can suppose $2l > 2k$ by interchanging the roles of b and b^{-1} if necessary.

If $2l > 2m$, let $2j = 2l$. If $2l < 2m$, then we can relabel X' according to $u_i \leftrightarrow u_{2m-i}$ and $v_i \leftrightarrow v_{2m-i}$ for $0 \le i \le 4m-1$. Now u_{4m-1} is adjacent to $v_{4m-2l-2}$ and $v_{4m-2k-2}$. Let $2j = \max\{4m-2l-2, 4m-2k-2\}$ so that $2j > 2m$.

If $2l = 2m$, then $bx^{-1} = a$. In this case consider the labelling $a^{-i} \leftrightarrow u_i$ and $xa^{-i} \leftrightarrow v_i$. Since $bx^{-1} = a$, u_{4m-1} is adjacent to v_{4m-2} by a b-edge, we let $2j = 4m-2$ in this case.

Thus, we can assume $2j > 2m$ in X'. If we find a Hamilton path P from u_0 to u_{2i} for $1 \le i \le 2m-1$ under this assumption, we shall have shown that X is Hamilton-connected.

CASE 1: $2i = 2j - 2m - 2$. Note that in this case $2j > 2m + 2$. A Hamilton path P from u_0 to u_{2i} is given by

$$\begin{aligned} P = {} & u_0v_0v_{4m-1}u_{2m-1}u_{2m}u_{2m+1}v_1v_2v_3 \cdots v_{2m-2}u_{2m-2}u_{2m-3}u_{2m-4} \\ & \cdots u_{2j-2m-1}v_{2j-1}v_{2j-2}v_{2j-3} \cdots v_{2m+2}u_{2m+2}u_{2m+3} \cdots u_{4m-2}v_{4m-2}v_{4m-3} \\ & \cdots v_{2j}u_{4m-1}v_{2m-1}v_{2m}v_{2m+1}u_1u_2 \cdots u_{2i}. \end{aligned}$$

CASE 2: $2i = 2j - 2m$. If $2j < 4m - 2$, a Hamilton path from u_0 to u_{2i} is given by

$$P = u_0 v_0 v_{4m-1} u_{2m-1} u_{2m} u_{2m+1} v_1 v_2 v_3 \cdots v_{2m-2} u_{2m-2} u_{2m-3} u_{2m-4} \cdots u_{2j-2m+1} v_{2j+1} v_{2j+2} v_{2j+3} \cdots v_{4m-2} u_{4m-2} u_{4m-3} u_{4m-4} \cdots u_{2m+2} v_{2m+2} v_{2m+3} \cdots v_{2j} u_{4m-1} v_{2m-1} v_{2m} v_{2m+1} u_1 u_2 \cdots u_{2i}.$$

If $2j = 4m - 2$, then $2i = 2m - 2$. We can choose P to be

$$P = u_0 v_0 v_{4m-1} u_{2m-1} u_{2m} v_{2m} v_{2m-1} v_{2m-2} \cdots v_1 u_{2m+1} u_{2m+2} \cdots u_{4m-1} v_{4m-2} v_{4m-3} \cdots v_{2m+1} u_1 u_2 \cdots u_{2m-2}.$$

CASE 3: $0 < 2i < 2j - 2m - 3$. In this case $2j > 2m + 4$. Let

$$P = u_0 v_0 v_{4m-1} u_{2m-1} u_{2m} u_{2m+1} v_1 v_2 v_3 \cdots v_{2i+2} u_{2i+2} u_{2i+1} v_{2i+2m+1} v_{2i+2m} v_{2i+2m-1} \cdots v_{2m+2} u_{2m+2} u_{2m+3} \cdots u_{2i+2m+2} v_{2i+2m+2} v_{2i+2m+3} u_{2i+3} u_{2i+4} v_{2i+4} v_{2i+3} u_{2i+2m+3} u_{2i+2m+4} v_{2i+2m+4} v_{2i+2m+5} u_{2i+5} \cdots u_{2j-2m-1} u_{2j-2m} u_{2j-2m+1} \cdots u_{2m-2} v_{2m-2} v_{2m-3} v_{2m-4} \cdots v_{2j-2m-1} u_{2j-1} u_{2j} u_{2j+1} \cdots u_{4m-2} v_{4m-2} v_{4m-3} \cdots v_{2j} u_{4m-1} v_{2m-1} v_{2m} v_{2m+1} u_1 u_2 \cdots u_{2i}.$$

In Case 3 note that when $2i = 2j - 2m - 4$, then the vertex u_{2i+3} at the end of line 2 is adjacent to u_{2m-2} of line 4. Thus, the segment in between is omitted.

CASE 4: $2j - 2m + 1 < 2i < 2m$. In this case, $2j < 4m - 2$. If $2i < 2m - 2$, then $2j \leq 4m - 6$. We build P in the following way:

$$P = u_0 v_0 v_{4m-1} u_{2m-1} u_{2m} u_{2m+1} v_1 v_2 \cdots v_{2j-2m} u_{2j-2m} u_{2j-2m-1} u_{2j-2m-2} \cdots u_1 v_{2m+1} v_{2m} v_{2m-1} u_{4m-1} v_{2j} v_{2j-1} v_{2j-2} \cdots v_{2m+2} u_{2m+2} u_{2m+3} u_{2m+4} \cdots u_{2j+1} v_{2j-2m+1} v_{2j-2m+2} u_{2j-2m+2} u_{2j-2m+1} v_{2j+1} v_{2j+2} u_{2j+2} u_{2j+3} v_{2j-2m+3} v_{2j-2m+4} u_{2j-2m+4} u_{2j-2m+3} v_{2j+3} v_{2j+4} u_{2j+4} u_{2j+5} v_{2j-2m+5} \cdots v_{2i-1} v_{2i} v_{2i+1} \cdots v_{2m-2} u_{2m-2} u_{2m-3} \cdots u_{2i+1} v_{2i+2m+1} v_{2i+2m+2} v_{2i+2m+3} \cdots v_{4m-2} u_{4m-2} u_{4m-3} \cdots u_{2i+2m} v_{2i+2m} v_{2i+2m-1} u_{2i-1} u_{2i}.$$

Next we suppose $2i = 2m - 2$. If $2j = 4m - 4$, then we use the path

$$P = u_0 u_1 u_2 \cdots u_{2m-3} v_{4m-3} v_{4m-2} v_{4m-1} v_0 v_1 \cdots v_{4m-4} u_{4m-1} u_{4m-2} \cdots u_{2m-2}.$$

If $2j = 2m + 2$, then we use the path

$$P = u_0 v_0 v_{4m-1} u_{2m-1} u_{2m} u_{2m+1} v_1 v_2 v_3 u_{2m+3} u_{2m+2} v_{2m+2} u_{4m-1} v_{2m-1} v_{2m} v_{2m+1} u_1 u_2 u_3 v_{2m+3} v_{2m+4} u_{2m+4} u_{2m+5} v_5 v_4 u_4 u_5 v_{2m+5} v_{2m+6} u_{2m+6} u_{2m+7} v_7 v_6 u_6 u_7 v_{2m+7} \cdots v_{4m-5} v_{4m-4} u_{4m-4} u_{4m-3} u_{4m-2} v_{4m-2} v_{4m-3} u_{2m-3} u_{2m-4} v_{2m-4} v_{2m-3} v_{2m-2} u_{2m-2}.$$

If $2m + 2 < 2j < 4m - 4$, then let

$$P = u_0 v_0 v_{4m-1} u_{2m-1} u_{2m} u_{2m+1} v_1 v_2 v_3 \cdots v_{2j-2m-1} u_{2j-1} u_{2j-2} \cdots u_{2m+2} v_{2m+2} v_{2m+3} v_{2m+4} \cdots v_{2j-1} u_{2j-2m-1} u_{2j-2m-2} \cdots u_1 v_{2m+1} v_{2m} v_{2m-1} u_{4m-1} v_{2j} u_{2j} u_{2j+1} v_{2j-2m+1} v_{2j-2m} u_{2j-2m} u_{2j-2m+1} v_{2j+1} v_{2j+2} u_{2j+2} u_{2j+3} v_{2j-2m+3} v_{2j-2m+2} u_{2j-2m+2} u_{2j-2m+3} v_{2j+3} \cdots v_{4m-5} v_{4m-4} u_{4m-4} u_{4m-3} u_{4m-2} v_{4m-2} v_{4m-3} u_{2m-3} u_{2m-4} v_{2m-4} v_{2m-3} v_{2m-2} u_{2m-2}.$$

CASE 5: $2i = 2m$. If $2m < 2j < 4m-2$, we construct a Hamilton path from u_0 to u_{2m} as follows:

$$\begin{aligned}P = {} & u_0 v_0 v_{4m-1} u_{2m-1} u_{2m-2} u_{2m-3} v_{4m-3} v_{4m-2} u_{4m-2} u_{4m-3} v_{2m-3} v_{2m-2} v_{2m-1} \\ & u_{4m-1} v_{2j} v_{2j+1} v_{2j+2} v_{2j+3} \cdots v_{4m-4} u_{4m-4} u_{4m-5} \cdots u_{2j-1} v_{2j-2m-1} v_{2j-2m} \\ & v_{2j-2m+1} \cdots v_{2m-4} u_{2m-4} u_{2m-5} u_{2m-6} \cdots u_{2j-2m-1} v_{2j-1} v_{2j-2} u_{2j-2} U15 u_{2j-3} \\ & v_{2j-2m-3} v_{2j-2m-2} u_{2j-2m-2} u_{2j-2m-3} v_{2j-3} \cdots v_{2m+1} v_{2m} u_{2m}.\end{aligned}$$

If $2j = 4m-2$, then we have the following Hamilton path from u_0 to u_{2m}:

$$\begin{aligned}P = {} & u_0 v_0 v_{4m-1} u_{2m-1} u_{2m-2} u_{2m-3} \cdots u_1 v_{2m+1} v_{2m+2} v_{2m+3} \cdots v_{4m-2} u_{4m-1} \\ & u_{4m-2} \cdots u_{2m+1} v_1 v_2 \cdots v_{2m} u_{2m}.\end{aligned}$$

CASE 6: $2m < 2i < 2j$. In this case let

$$\begin{aligned}P = {} & u_0 v_0 v_{4m-1} u_{2m-1} u_{2m} u_{2m+1} v_1 v_2 \cdots v_{2i-2m} u_{2i-2m} u_{2i-2m-1} u_{2i-2m-2} \cdots u_1 \\ & v_{2m+1} v_{2m} v_{2m-1} u_{4m-1} v_{2j} v_{2j+1} v_{2j+2} \cdots v_{4m-2} u_{4m-2} u_{4m-3} u_{4m-4} \cdots u_{2j-1} \\ & v_{2j-2m-1} v_{2j-2m} v_{2j-2m+1} \cdots v_{2m-2} u_{2m-2} u_{2m-3} \cdots u_{2j-2m-1} v_{2j-1} v_{2j-2} \\ & u_{2j-2} u_{2j-3} v_{2j-2m-3} v_{2j-2m-2} u_{2j-2m-2} u_{2j-2m-3} v_{2j-3} v_{2j-4} u_{2j-4} u_{2j-5} \\ & v_{2j-2m-5} v_{2j-2m-4} u_{2j-2m-4} u_{2j-2m-5} v_{2j-5} \cdots v_{2i+1} v_{2i} v_{2i-1} v_{2i-2} \cdots v_{2m+2} \\ & u_{2m+2} u_{2m+3} \cdots u_{2i}.\end{aligned}$$

CASE 7: $2i = 2j$. If $2m+2 < 2j \le 4m-6$, then use

$$\begin{aligned}P = {} & u_0 v_0 v_{4m-1} u_{2m-1} u_{2m} u_{2m+1} v_1 v_2 \cdots v_{2j-2m-2} u_{2j-2m-2} u_{2j-2m-3} u_{2j-2m-4} \\ & \cdots u_1 v_{2m+1} v_{2m} v_{2m-1} u_{4m-1} v_{2j} v_{2j+1} v_{2j+2} v_{2j+3} \cdots v_{4m-5} u_{2m-6} u_{2m-7} \\ & u_{2m-3} \cdots u_{2j-2m-1} v_{2j-1} v_{2j-2} v_{2j-3} \cdots v_{2m+2} u_{2m+2} u_{2m+3} \cdots u_{2j-1} \\ & v_{2j-2m-1} v_{2j-2m} v_{2j-2m+1} \cdots v_{2m-4} u_{2m-4} u_{2m-3} u_{2m-2} v_{2m-2} v_{2m-3} u_{4m-3} \\ & u_{4m-2} v_{4m-2} v_{4m-3} v_{4m-4} u_{4m-4} u_{4m-5} u_{4m-6} \cdots u_{2j}.\end{aligned}$$

If $2j = 2m+2$, then let

$$\begin{aligned}P = {} & u_0 v_0 v_{4m-1} u_{2m-1} u_{2m-2} u_{2m-3} \cdots u_1 v_{2m+1} v_{2m} u_{2m} u_{2m+1} v_1 v_2 \cdots v_{2m-1} \\ & u_{4m-1} v_{2m+2} v_{2m+3} \cdots v_{4m-2} u_{4m-2} u_{4m-3} \cdots u_{2m+2}.\end{aligned}$$

If $2j = 4m-2$, then let

$$\begin{aligned}P = {} & u_0 v_0 v_{4m-1} u_{2m-1} u_{2m} v_{2m} v_{2m-1} u_{4m-1} v_{4m-2} v_{4m-3} \cdots v_{2m+1} u_1 u_2 \cdots u_{2m-2} \\ & v_{2m-2} v_{2m-3} \cdots v_1 u_{2m+1} u_{2m+2} \cdots u_{4m-2}.\end{aligned}$$

If $2j = 4m-4$, then use

$$\begin{aligned}P = {} & u_0 v_0 v_{4m-1} v_{4m-2} v_{4m-3} u_{2m-3} u_{2m-2} \cdots u_{2m+1} v_1 v_2 \cdots v_{2m-4} u_{2m-4} u_{2m-5} \\ & \cdots u_1 v_{2m+1} v_{2m} \cdots v_{2m-3} u_{4m-3} u_{4m-2} u_{4m-1} v_{4m-4} v_{4m-5} \cdots v_{2m+2} u_{2m+2} \\ & u_{2m+3} \cdots u_{4m-4}.\end{aligned}$$

CASE 8: $2i > 2j > 2m$. In this case we use

$$\begin{aligned} P = {} & u_0 v_0 v_{4m-1} u_{2m-1} u_{2m} u_{2m+1} v_1 v_2 \cdots v_{2j-2m} u_{2j-2m} u_{2j-2m-1} u_{2j-2m-2} \\ & \cdots u_1 v_{2m+1} v_{2m} v_{2m-1} u_{4m-1} v_{2j} v_{2j-1} v_{2j-2} v_{2j-3} \cdots v_{2m+2} u_{2m+2} u_{2m+3} \\ & u_{2m+4} \cdots u_{2j+1} v_{2j-2m+1} v_{2j-2m+2} u_{2j-2m+2} u_{2j-2m+1} v_{2j+1} v_{2j+2} u_{2j+2} u_{2j+3} \\ & v_{2j-2m+3} v_{2j-2m+4} u_{2j-2m+4} u_{2j-2m+3} v_{2j+3} v_{2j+4} u_{2j+4} u_{2j+5} v_{2j-2m+5} \\ & \cdots v_{2i-2m-1} v_{2i-2m} v_{2i-2m+1} v_{2i-2m+2} \cdots v_{2m-2} u_{2m-2} \cdots u_{2i-2m-1} \\ & v_{2i-1} v_{2i} v_{2i+1} \cdots v_{4m-2} u_{4m-2} u_{4m-3} u_{4m-4} \cdots u_{2i}. \end{aligned}$$

This completes the proof. □

4. MAIN THEOREM

In this section we state and prove the main theorem of the paper.

THEOREM 4.1. *Let $X = X(G; S)$ be a connected Cayley graph on a Hamiltonian group G. If* $\mathrm{val}(X) \geq 3$*, then X is Hamilton-connected if X is not bipartite or X is Hamilton-laceable if X is bipartite.*

PROOF. Since $G = Q \times U \times V$, as mentioned at the beginning of the last section, the smallest Hamiltonian group is the quaternion group Q, which has eight elements. Every generating set S of Q contains at least two generators of order four, which means $X(Q; S)$ contains the complete bipartite graph, with each part of cardinality four, as a spanning subgraph. It is easy to check $X(Q; S)$ is Hamilton-laceable if it is bipartite or Hamilton-connected if it is not bipartite. In the following we consider G with order greater than eight and apply induction on the order of G.

First we assume that S is minimal in the sense that for every $a \in S$, $S \setminus \{a, a^{-1}\}$ is not a generating set of G. By Corollary 3.3, we can suppose that every element of S is of even order. Let $a \in S$ be an element not in the centre of G. Let $S' = S \setminus \{a, a^{-1}\}$ and $X' = X(\langle S' \rangle; S')$. Since $\langle S' \rangle$ is abelian, the Chen–Quimpo theorem implies that X' falls into one of the following three cases:

(i) X' is Hamilton-connected;
(ii) X' is a cycle, or
(iii) X' is bipartite and Hamilton-laceable.

CASE 1: X' is Hamilton-connected. By considering the quotient graph on $G/\langle S' \rangle$ generated by $\bar{a}$ and applying Lemma 3.2, we know that X is Hamilton-connected.

CASE 2: X' is a cycle. In this case, $S = \{a, a^{-1}, b, b^{-1}\}$ with $o(a) = 4m$ and $o(b) = 4k$ for some positive integers m and k. If we write $a = (q_1, x_1, y_1)$ and $b = (q_2, x_2, y_2)$, according to the direct product $G = Q \times U \times V$, we know that $ab^2 = b^2 a$ and, thus, $a^i = b^j$ implies both i and j are even. This implies that $X(G/\langle b \rangle, \{\bar{a}, \bar{a}^{-1}\})$ is of even order. Let X_0 denote the subgraph generated by $\langle b \rangle \cup a\langle b \rangle$. Similarly to the proof for $X' \cong GP(4m, 2m + 1)$ in Lemma 3.5, $X_0 \cong GP(4k, 2k + 1) \cong GP(4k, 2k - 1)$ which is Hamilton-laceable by Theorem 2.5.

Let X_j denote the subgraph generated by $a^{2j}\langle b \rangle \cup a^{2j+1}\langle b \rangle$. Since left multiplication by a^{2j} is an isomorphism of X_0 to X_j, $X_j \cong GP(4k, 2k + 1) \cong GP(4k, 2k - 1)$. Therefore, X is bipartite because $a^i = b^j$ implies j is even.

Let $u \in A$ and $v \in B$, where A and B are the two bipartition sets. Without loss of generality, let $u \in X_i$ and $v \in X_j$ with $i \leq j$. When $i < j$, choose $\{v_i, v_{i+1}, \ldots, v_{j-1}\} \subseteq B$ and $\{u_{i+1}, u_{i+2}, \ldots, u_j\} \subseteq A$ so that $v_k \in X_k$, $u_s \in X_s$ and v_k is adjacent to u_{k+1}, $i \leq k \leq j-1$. Let P_i be a Hamilton path in X_i from u to v_i, P_j be a Hamilton path in X_j from v to u_j and P_{i+r} be a Hamilton path from u_{r+i} to v_{r+i} in X_{i+r} for $1 \leq r \leq j-i-1$. When $j = i$, simply let P_i be a Hamilton path from u to v in X_i. It is easy to see that $P' = P_i \cup P_{i+1} \cup \cdots \cup P_j \cup \{u_r v_{r-1} : i+1 \leq r \leq j\}$ is a path from u to v containing all the vertices of $X_i \cup X_{i+1} \cup \cdots \cup X_j$.

Since $o(b) \geq 4$, there is an edge xy in both P' and $a^{2j+1}\langle b\rangle$. Let x_{j+2} and y_{j+2} be the neighbours of x and y, respectively, in $a^{2j+2}\langle b\rangle$, where we use $2j+2 = 0$ if $j = o(\bar{a})/2 - 1$. Let P_{j+2} be a Hamilton path in X_{j+2} joining x_{j+2} and y_{j+2}. Removing the edge xy from P' and adding the edges of P_{j+2} together with xx_{j+2} and yy_{j+2} produces a path from u to v containing all the vertices of $X_i \cup X_{i+1} \cdots X_j \cup X_{j+1}$. Since $X_0, X_1, \ldots, X_{0(\bar{a})/2-1}$ are joined in a cycle, we repeat this process until we have a path P'' from u to v using all the vertices of X. Therefore, X is Hamilton-laceable.

CASE 3: X' is bipartite and Hamilton-laceable. In this case, we prove that X is bipartite too. It follows from Lemma 3.4 that X is Hamilton-laceable.

Because G is non-abelian and $|S'| > 2$, there exist $b, c \in S'$ such that $c \notin \{b, b^{-1}\}$, b satisfies $ab \neq ba$ and $o(b) > 2$. Since $\langle b\rangle \lhd \langle S'\rangle$, $\langle S'\rangle$ can be partitioned into the union of left cosets of $\langle b\rangle$ each of which corresponds to a cycle of length $o(b)$.

We first prove that the subgraph generated by $\langle S'\rangle \cup a\langle S'\rangle$ is bipartite. Now $\langle b\rangle \lhd G$ implies $a\langle b\rangle = \langle b\rangle a$. So there exists an integer i such that $ba = ab^i$, that is $b^i = a^{-1}ba$. Therefore, $o(b^i) = o(b)$ which implies i must be odd. So the subgraph induced by $\langle b\rangle \cup a\langle b\rangle$ is bipartite.

Let A_i and B_i be the bipartition sets of $a^i\langle S'\rangle$, $0 \leq i < o(\bar{a})$. Suppose $1 \in A_0$. Choose $x \in \langle S'\rangle$ so that $x \in A_0$. Then x_L, left multiplication by x, satisfies $x_L(A_0) = A_0$ and $x_L(B_0) = B_0$. Meanwhile, $a_L(A_i) = B_{i+1}$ and $a_L(B_i) = A_{i+1}$ for $0 \leq i \leq o(\bar{a})$. Now $xa = a^j x$, for some j, and the latter implies $xax^{-1} = a^j$ so that j is odd as in the above argument. Thus, xa is in the same bipartition set as ax, which is in the same bipartition set as a. Hence, $x_L : A_1 \to A_1$ and $x_L : B_1 \to B_1$. Similarly, $x_L : A_i \to A_i$ and $x_L : B_i \to B_i$ for all i. We then see that $\langle S'\rangle \cup a\langle S'\rangle \cup \cdots \cup a^{o(\bar{a})-1}\langle S'\rangle$ together with the a-edges of the form x, xa from $a^i\langle S'\rangle$ to $a^{i+1}\langle S'\rangle$, $0 \leq i \leq o(\bar{a}) - 1$, is bipartite. It remains to show that the edges from $a^{o(\bar{a})}\langle S'\rangle$ to $\langle S'\rangle$ respect the bipartition sets.

We claim that there exist j and k such that $a^j = b^k$ with $j < o(a)$ in which case $j = o(\bar{a})$. If not, $\langle a\rangle \cap \langle b\rangle = 1$. Since $\langle a\rangle \lhd \langle\{a, b\}\rangle$ and $\langle b\rangle \lhd \langle\{a, b\}\rangle$, it then would follow that $\langle a, b\rangle \cong \langle a\rangle \times \langle b\rangle$ implying that $a = b$, contradicting our choices of a and b. Since $ab^k = b^k a$, k has to be even. Similarly, j has to be even. Therefore, X is bipartite.

Now we consider the case where S is not minimal. If there exists a subset T of S such that $\langle T\rangle = G$ and $X(G; T)$ is not bipartite, or both $X(G; T)$ and $X(G; S)$ are bipartite, then we are done by the above discussion. So we may assume that, for any subset T of S with $\langle T\rangle = G$, $X(G; T)$ is bipartite but $X(G; S)$ is not bipartite.

Let a and x be two elements in S such that $ax \neq xa$. First assume that $S \setminus \{a, a^{-1}\}$ is not a generating set of G. Then if $X(\langle S \setminus \{a, a^{-1}\}\rangle; S \setminus \{a, a^{-1}\})$ is not bipartite it is Hamilton-connected by induction. Hence, by Lemma 3.2, $X(G; S)$ is Hamilton-connected. If $X(\langle S \setminus \{a, a^{-1}\}\rangle; S \setminus \{a, a^{-1}\})$ is bipartite, then $X(G; S)$ is bipartite by the proof of Case 3 above, which contradicts our hypothesis. Therefore, we may assume $S \setminus \{a, a^{-1}\}$ is a generating set of G and $X(G; S \setminus \{a, a^{-1}\})$ is bipartite.

Let $Y = X(G; S \setminus \{a, a^{-1}\})$ and A and B denote the two bipartition sets of Y. Since X is not bipartite and left multiplication by a is an isomorphism of X, each a-edge is contained either entirely in A or entirely in B. Also, since G is a Hamiltonian group, if there is an a-edge between $u\langle x\rangle$ and $v\langle x\rangle$, $u, v \in G$, then there is a matching consisting of a-edges between $u\langle x\rangle$ and $v\langle x\rangle$.

Let $Z = X(G/\langle x\rangle; \overline{S \setminus \{a, a^{-1}, x, x^{-1}\}})$. The graph Z is connected because Y is connected. Since x and a do not commute, we know that there are an even number of cosets of $\langle x\rangle$, that is, Z has even order.

In the following, we assume that the vertices of Y have been partitioned into cosets of $\langle x\rangle$, that is, the partition sets correspond to vertices of Z. Let w be any vertex in A. Because X is vertex transitive, if we can prove that there is a Hamilton path from w to each vertex v in A, then, since Y is Hamilton-laceable, X is Hamilton-connected.

We now consider the case where there are at least four cosets of $\langle x\rangle$, saving the case of two cosets until the end. First we assume that w and v are in different cosets of $\langle x\rangle$. Let $\overline{C} = \overline{w_0 w_1} \cdots \overline{w_\ell}$ be a Hamilton cycle in Z. Such a cycle exists by the Chen–Quimpo theorem. Even though no $\overline{a}$-edges are used in $\overline{C}$, we are interested in which $\langle x\rangle$ cosets are joined by a-edges since they are the edges making X non-bipartite. Let $\overline{w_0}$ and $\overline{w_j}$ be the cosets of $\langle x\rangle$ containing w and v, respectively. Since $\overline{C}$ is a cycle, we may assume that the coset $\overline{w_i}$ joined to $\overline{w_j}$ by a-edges in X satisfies $0 \le i < j$.

The a-edges between $\overline{w_i}$ and $\overline{w_j}$ join vertices of A to vertices of A and vertices of B to vertices of B. Thus, if we view these two $o(x)$-cycles together with the a-edges joining them as a generalized Petersen graph H (recall it is isomorphic to $GP(o(x), o(x)/2 - 1)$), the vertices of A in $\overline{w_j}$ are in the opposite bipartition set of the A vertices of $\overline{w_i}$. Therefore, by Theorem 2.5, there is a Hamilton path P' in H from any vertex of A in $\overline{w_i}$ to v.

We now develop a useful property of successive cosets in $\overline{C}$. Suppose that the cosets $\overline{w_k}$ and $\overline{w_{k+1}}$ are joined by b-edges for some $b \in S$. If $bx = xb$, then the subgraph spanned by these two $o(x)$-cycles and the b-edges is isomorphic to a Cayley graph on an abelian group. Thus, it is Hamilton-laceable by the Chen–Quimpo theorem. If $bx \ne xb$, then it is isomorphic to $GP(o(x), o(x)/2 - 1)$, which is Hamilton-laceable by Theorem 2.5.

If $j - i$ is odd, then choose an edge xy of P' from the $o(x)$-cycle corresponding to $\overline{w_i}$. Let the neighbours of x and y in $\overline{w_{i+1}}$ be x' and y', respectively. Then there is a Hamilton path from x' to y' using all the vertices of $\overline{w_{i+1}}$ and $\overline{w_{i+2}}$. Let this path replace the edge xy. Continue in this way until all vertices between $\overline{w_i}$ and $\overline{w_j}$ have been used. Since $j - i$ is odd this is possible.

Start the required path P at w by travelling around the $o(x)$-cycle at $\overline{w_0}$ until reaching the vertex z of B next to w. Then use the edge from z to a vertex z' of A on the $o(x)$-cycle at $\overline{w_1}$. Travel around this cycle until leaving at the vertex of B next to z', where $\overline{w_1}$ was entered. Continue in this way until entering $\overline{w_i}$ at a vertex of A. Then add on P' described above. This uses all vertices from $\overline{w_0}$ through $\overline{w_j}$. If there are vertices of $\overline{C}$ not used, there must be an even number left over. Add the vertices of the corresponding $o(x)$-cycles starting with an edge of the $o(x)$-cycle at $\overline{w_j}$ until reaching $\overline{w_\ell}$. The procedure just described works for any even number of cosets of $\langle x\rangle$.

If $j - i \ge 2$ is even, there is a path using all the vertices of the cosets $\overline{w_i} \cup \overline{w_{i+1}}$ joining any vertex of A in $\overline{w_i}$ to any vertex of B in the same coset. Thus, start a path at v in $\overline{w_j}$ travelling around the $o(x)$-cycle until reaching the vertex z of B adjacent to v. Then use the a-edge from z to its neighbour $z' \in \overline{w_i}$. Note that z' is in B. Continue this path by taking a Hamilton path from z' to a vertex of A, to be chosen later, in $\overline{w_i}$ using all vertices of the cosets $\overline{w_i}$ and $\overline{w_{i+1}}$. If $j - i > 2$, then do as before to include all vertices from cosets between $\overline{w_i}$ and $\overline{w_j}$. We now have a path P' from v to any vertex of A in $\overline{w_i}$ using all vertices of $\overline{w_i} \cup \overline{w_{i+1}} \cup \cdots \cup \overline{w_j}$.

Next find a path from w in $\overline{w_0}$ to a vertex x of B also in $\overline{w_0}$ using all vertices of $\overline{w_0} \cup \overline{w_\ell}$. This can then be extended to use all vertices of $\overline{w_{j+1}} \cup \cdots \cup \overline{w_{\ell-1}}$ as described previously.

Finally, from the vertex x in $\overline{w_0}$ extend the path through successive cosets until reaching a vertex in A in $\overline{w_i}$. This is the vertex to be chosen for the end of P'. We now have a Hamilton path joining v and w.

This leaves only the case where $\langle x \rangle$ has two cosets. In this case, since $G = \langle x, a \rangle$, $ax \neq xa$ and $X(G; S)$ is not bipartite, there exists an element $b \in S$ such that $b\langle x^2 \rangle = ax\langle x^2 \rangle$. Note that $xb \neq bx$. If $o(b) < |G|/2$, that is, $\langle b \rangle$ has more than two cosets in G, we can consider $\langle b \rangle$ instead of $\langle x \rangle$. We are done by considering a previous case. So we suppose $o(b) = |G|/2$. Similarly, we suppose $o(a) = |G|/2$. Therefore, $X(G; S)$ contains the graph in Figure 1 as a spanning subgraph. By Lemma 3.5, $X(G; S)$ is Hamilton-connected. This completes the proof. □

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